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APPLICATION OF THE FRACTIONAL ORDER CAUCHY PROBLEM IN POPULATION DYNAMICS AND ITS EVALUATION BASED ON REAL DATA

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Abstract
This article studies the application of the Cauchy problem in population dynamics. The classical population model is usually represented by a first-order ordinary differential equation. However, in real biological processes, population growth may depend not only on the current state but also on conditions in previous periods. Therefore, in this article the classical Cauchy problem is generalized by means of a fractional-order derivative in the Caputo sense. In particular, models of orders and are considered and compared with the classical model. Experimental data on the population of Paramecium caudatum are used to evaluate how close the models are to the real process. The results of the analysis show that fractional-order models provide results closer to real data than the classical model.

Keywords: Cauchy problem, population dynamics, fractional-order derivative, Caputo derivative, Mittag-Leffler function, memory effect, Paramecium caudatum, model error.

Introduction

Population dynamics is one of the important areas of biology, ecology, and mathematical modeling. By studying the change over time in the number of living organisms inhabiting a given area or environment, it is possible to evaluate their future state [7]. Such processes are usually described using differential equations and initial conditions.

If the value of a population at the initial time is known and the state at subsequent times is determined on the basis of this value, such a problem is written in the form of a Cauchy problem. In the classical case, population growth is represented by a first-order ordinary differential equation. However, in real biological processes, the current state of a population is related to the amount of food in previous periods, environmental conditions, reproductive delay, disease, and other factors. To take such factors into account, a memory effect is needed. This is reflected in the fractional-order derivative. Therefore, when constructing a mathematical model of such processes, we try to describe the process more realistically by using fractional-order derivatives.

Classical population model

The simplest model of population dynamics is written by the following Cauchy problem [5]:

$$P'(t) = rP(t), \qquad (1)$$

$$P(0) = P_0. \qquad (2)$$

Here $P(t)$ is the population size at time t, P_0 is the initial population size, and r is the growth coefficient.

In this model, the condition $P(0) = P_0$ is the Cauchy condition. The equation means that the growth rate of the population is proportional to its current size.

The solution of the classical Cauchy problem is written as follows:

$$P(t) = P_0 e^{rt}. \qquad (3)$$

If $P_0 = 100$ and $r = 0.1$, then the model takes the following form:

$$P'(t) = 0.1P(t), \qquad P(0) = 100.$$

Its solution is:

$$P(t) = 100e^{0.1t}.$$

If $t = 5$, then:

$$P(5) = 100e^{0.5} \approx 164.9.$$

Thus, according to the classical model, after 5 time units the population reaches approximately 165 individuals.

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Limitation of the classical model

Although the classical model is simple and convenient, it considers population growth to depend only on the current population size. That is, in the equation $P'(t) = rP(t)$, the population state in previous periods is not directly taken into account.

In fact, in real biological processes, population growth depends on the following factors:

Factor	**Effect on the population**
Food supply	If food is insufficient, the population grows slowly or decreases.
Climate change	Hot or cold weather affects the number of animals, plants, and microorganisms.
Diseases	Disease reduces the population size or slows down its growth rate.
Biological delay	It causes a time lag in the processes of birth and growth.
Effect of previous periods	It affects current growth.

The above factors are important parameters in describing a real population process. These parameters cannot be expressed by a simple classical model. In such cases, it is appropriate to use a model constructed with the help of a fractional-order derivative.

Fractional-order Caputo derivative. To take into account the memory effect in the population process, we use the fractional-order derivative in the Caputo sense. If $0 < \alpha < 1$, the Caputo derivative is defined as follows:

$$ {}^{C}D_{0+}^{\alpha} P(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{P'(\tau)}{(t-\tau)^{\alpha}}, d\tau. \qquad (4)$$

Here $\Gamma(\cdot)$ is the Gamma function, and α denotes the fractional order of the derivative.

It follows from formula (4) that ${}^{C}D_{0+}^{\alpha} P(t)$ depends not only on the value at time t, but also on all previous states in the interval from 0 to t. Therefore, the Caputo derivative is convenient for modeling processes with a memory effect.

Fractional-order Cauchy problem for population dynamics. In population dynamics, the fractional-order Cauchy problem is written as follows:

$$ {}^{C}D_{0+}^{\alpha} P(t) = rP(t), \qquad (5)$$

$$ P(0) = P_0, \qquad (6)$$

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Here $P(t)$ is the population size at time t, P_0 is the initial population size, r is the growth coefficient, and α, where $0<\alpha\leq 1$, denotes the order of the fractional derivative.

If $\alpha=1$ in equation (5), the model returns to the classical case:

$$ {}^{C}D_{0+}^{1}P(t)=P'(t). \qquad (7)$$

Therefore, for $\alpha=1$, the following classical model is obtained: $P'(t)=rP(t), \quad P(0)=P_0$.

The solution of the fractional-order Cauchy problem is expressed through the Mittag-Leffler function:

$$P(t)=P_0E_\alpha(rt^\alpha). \qquad (8)$$

In equality (8), $E_\alpha(z)$ is the Mittag-Leffler function and is defined as follows:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}. \qquad (9)$$

If $\alpha=1$, then $E_1(z)=e^z$. From the above, we conclude that the classical model is a special case of the fractional-order model.

If $\alpha=\frac{1}{2}$, then the model is written as follows:

$$ {}^{C}D_{0+}^{\frac{1}{2}}P(t)=rP(t), \qquad P(0)=P_0. \qquad (10)$$

From equation (4), according to the definition of the Caputo derivative:

$$ {}^{C}D_{0+}^{\frac{1}{2}}P(t)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\frac{P'(\tau)}{(t-\tau)^{\frac{1}{2}}},d\tau. \qquad (11)$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \qquad (12)$$

Taking this into account, we obtain the following equality:

$$ {}^{C}D_{0+}^{\frac{1}{2}}P(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{P'(\tau)}{\sqrt{t-\tau}},d\tau. \qquad (13)$$

Substituting equality (12) into equation (10), the Cauchy problem takes the following form:

$$\frac{1}{\sqrt{\pi}}\int_0^t\frac{P'(\tau)}{\sqrt{t-\tau}},d\tau=rP(t) \qquad (14)$$

$$P(0)=P_0. \qquad (15)$$

Using the Mittag-Leffler function, the solution of the Cauchy problem (11)-(12) can be expressed in the following form:

$$P(t) = P_0 E_{\frac{1}{2}}(rt^{\frac{1}{2}}).$$

If $\alpha = \frac{1}{3}$, then the model is written as follows:

$$^{C}D_{0+}^{\frac{1}{3}} P(t) = rP(t), \qquad P(0) = P_0.$$

According to the definition of the Caputo derivative:

$$^{C}D_{0+}^{\frac{1}{3}} P(t) = \frac{1}{\Gamma\left(1-\frac{1}{3}\right)} \int_0^t \frac{P'(\tau)}{(t-\tau)^{\frac{1}{3}}}, d\tau.$$

That is:

$$^{C}D_{0+}^{\frac{1}{3}} P(t) = \frac{1}{\Gamma\left(\frac{2}{3}\right)} \int_0^t \frac{P'(\tau)}{(t-\tau)^{\frac{1}{3}}}, d\tau.$$

Therefore, the population model of order $\frac{1}{3}$ has the following integral form:

$$\frac{1}{\Gamma\left(\frac{2}{3}\right)} \int_0^t \frac{P'(\tau)}{(t-\tau)^{\frac{1}{3}}}, d\tau = rP(t), \qquad P(0) = P_0.$$

The solution of this Cauchy problem is:

$$P(t) = P_0 E_{\frac{1}{3}}(rt^{\frac{1}{3}}).$$

Real population data

To evaluate the closeness of the theoretical model to the real process, we use experimental data related to a population. The object of the study is a unicellular organism living in an aquatic environment, which is convenient for studying population dynamics under laboratory conditions. The data presented in Gause's experiments [1] are widely used to analyze the growth of organism species in separate and mixed environments.

The following table presents experimental data on the growth of Paramecium caudatum in a separate environment. The population amount $cells / mL$ is expressed in cells/mL, that is, as the number of cells in 1 mL. The values in the table represent observations from day 0 to day 16.

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Day	0	2	4	6	8	10	12	14	16
$P_{\text{real}}(t)$ $cells/mL$	2	8	30	48	58	60	58	58	60

As can be seen from this table, the population grows rapidly at first. On day 0, the population was $2\, cells/mL$; by day 4 it reached $30\, cells/mL$, and by day 8 it reached $58\, cells/mL$. In the following days, the population stabilizes around $58-60\, cells/mL$.

This indicates that population growth in a real process does not continue indefinitely, but remains around a certain value due to environmental capacity and limited resources.

Construction of models based on real data. For this population, the initial condition is taken as follows:

$P(0)=2.$

According to the above Cauchy problem, the classical model is written as:

$P'(t)=rP(t), \qquad P(0)=2$

It is written in this form.

Its solution is expressed as:

$P(t)=2e^{rt}$

In this way we write it. The fractional model of order $\frac{1}{2}$ is:

$^{C}D_{0+}^{\frac{1}{2}}P(t)=rP(t), \qquad P(0)=2$

It is written in this form. Its solution is determined as follows:

$P(t)=2E_{\frac{1}{2}}(rt^{\frac{1}{2}}).$

$\alpha=\frac{1}{3}$:

$^{C}D_{0+}^{\frac{1}{3}}P(t)=rP(t), \qquad P(0)=2$

It is written in this form, and its solution is:

$P(t)=2E_{\frac{1}{3}}(rt^{\frac{1}{3}}).$

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Here, the value of r is not chosen arbitrarily. For each model, the parameter r is selected so that it best fits the real experimental data. That is, the difference between the model values and the actual values should be as small as possible.

Comparison of the models with real data. In the calculations, for each model the parameter r was computed based on the real data. The following results were obtained:

Day	**Actual value**	**Classical model**	$\alpha = \frac{1}{2}$	$\alpha = \frac{1}{3}$
0	2	2.00	2.00	2.00
2	8	3.18	4.72	6.05
4	30	5.07	7.60	9.62
6	48	8.07	11.64	14.32
8	58	12.84	17.45	20.70
10	60	20.44	25.89	29.48
12	58	32.54	38.18	41.61
14	58	51.80	56.09	58.41
16	60	82.46	82.22	81.69

As can be seen from this table, the classical model cannot adequately describe the rapid real growth of the population at the initial stage. At later stages, it overestimates the population. Fractional-order models give results closer to the real values than the classical model. In particular, the model of order $\alpha = \frac{1}{3}$ approaches the real data better at many points than the classical model and the model of order $\alpha = \frac{1}{2}$.

In the fractional-order model, the parameter α represents the memory effect in the population process. If the value of α is close to 1, the model approaches classical exponential growth. Conversely, as the value of α decreases, the influence of previous periods becomes stronger and population growth slows down. Therefore, in the case $\alpha = \frac{1}{3}$, the population grows more slowly than in the case $\alpha = \frac{1}{2}$; in the cases $\alpha = \frac{1}{4}$ or $\alpha = \frac{1}{5}$, growth becomes even slower.

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Error analysis. The error for each model is calculated as follows:

$e_i = \left| P_{\text{real}}(t_i) - P_{\text{model}}(t_i) \right|.$

Here $P_{\text{real}}(t_i)$ is the actual value observed in the experiment, and $P_{\text{model}}(t_i)$ is the value calculated by the model.

The calculated errors are given in the following table:

Day	Classical model error	$\alpha = \frac{1}{2}$ model error	$\alpha = \frac{1}{3}$ model error
0	0.00	0.00	0.00
2	4.82	3.28	1.95
4	24.93	22.40	20.38
6	39.93	36.36	33.68
8	45.16	40.55	37.30
10	39.56	34.11	30.52
12	25.46	19.82	16.39
14	6.20	1.91	0.41
16	22.46	22.22	21.69

As can be seen from the table, the errors of the classical model are generally large. In the model of order $\alpha = \frac{1}{2}$, the errors decrease. The model of order $\alpha = \frac{1}{3}$ gives the smallest total error for these real data.

General accuracy indicators

To evaluate the overall accuracy of the models, we use the mean absolute error, the root mean square error, and the mean absolute percentage error.

The mean absolute error is defined as follows:

$$MAE = \frac{1}{n} \sum_{i=1}^{n} \left| P_{\text{real}}(t_i) - P_{\text{model}}(t_i) \right|.$$

The root mean square error is defined as follows:

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(P_{\text{real}}(t_i) - P_{\text{model}}(t_i) \right)^2}.$$

The mean absolute percentage error is defined as follows:

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$$MAPE = \frac{100\%}{n}\sum_{i=1}^{n}\left|\frac{P_{\text{real}}(t_i) - P_{\text{model}}(t_i)}{P_{\text{real}}(t_i)}\right|.$$

The calculation results are as follows:

Model	Mean error	Root mean square error	Mean error (%)
Classical model, $\alpha = 1$	23.17	27.96	51.37%
Fractional model, $\alpha = \frac{1}{2}$	20.07	24.79	43.63%
Fractional model, $\alpha = \frac{1}{3}$	18.03	22.65	38.08%

According to these results, among the three considered models for the data of the real experimental object, the fractional model of order $\alpha = \frac{1}{3}$ gave the smallest error.

That is:

$$MAE_{\alpha=\frac{1}{3}} < MAE_{\alpha=\frac{1}{2}} < MAE_{\alpha=1}.$$

Also:

$$RMSE_{\alpha=\frac{1}{3}} < RMSE_{\alpha=\frac{1}{2}} < RMSE_{\alpha=1}.$$

And:

$$MAPE_{\alpha=\frac{1}{3}} < MAPE_{\alpha=\frac{1}{2}} < MAPE_{\alpha=1}.$$

Thus, the fractional model of order $\alpha = \frac{1}{3}$ was evaluated as the model closest to these real data.

Degree of closeness of the models to real data

Summarizing the results, the degree of closeness of the models to real data can be ranked as follows:

Rank	Model	Closeness to real data
1st place	$\alpha = \frac{1}{3}$	Closest
2nd place	$\alpha = \frac{1}{2}$	Moderately close
3rd place	classical model, $\alpha = 1$	Farthest

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Thus, for these experimental data, fractional-order models gave more accurate results than the classical model. In particular, the model of order $\alpha = \frac{1}{3}$ described the real population changes better.

Conclusion

1. According to the experiment, the population grows rapidly at first and then stabilizes around $58-60\ cells/mL$. This shows that population growth does not continue indefinitely, but is limited by environmental capacity and resources.
2. The classical model $P'(t) = rP(t)$ describes the population as a continuously growing process. Therefore, it cannot fully describe the stabilization stage.
3. The fractional-order model takes into account the influence of previous periods in population growth. That is, the current growth of the population is considered to depend not only on the current value but also on previous states. This is natural for biological processes.

In this analysis, the fact that the $\alpha = \frac{1}{3}$ model gives the best result shows that the memory effect is significant in the population process. However, the simple fractional exponential model also cannot fully describe the stabilization stage of the population.

Therefore, to construct a more accurate model, a fractional-order logistic model can be used:

$$ {}^{C}D_{0+}^{\alpha} P(t) = rP(t)\left(1 - \frac{P(t)}{K}\right). $$

Here K is the environmental capacity. Based on the real data, K is approximately taken around $60\ cells/mL$.

This model describes both the initial growth of the population and its subsequent stabilization more accurately.

In this article, the application of the Cauchy problem in population dynamics and its generalization by means of a fractional-order derivative were considered. The classical population model:

$$ P'(t) = rP(t), \qquad P(0) = P_0 $$

was written in this form. The solution of this model:

$$ P(t) = P_0 e^{rt} $$

was shown.

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Then the classical model was generalized using the fractional-order derivative in the Caputo sense:

$$ {}^{C}D_{0+}^{\alpha}P(t) = rP(t), \qquad P(0) = P_0, \qquad 0 < \alpha \le 1. $$

The solution of this problem was written through the Mittag-Leffler function as follows:

$$ P(t) = P_0 E_\alpha (rt^\alpha). $$

In particular, models of orders $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$ were considered separately. In the theoretical example, it was shown that as the value of α decreases, population growth slows down. This is explained by the strengthening of the memory effect in the population process.

In the practical part of the article, the classical model and the models with $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$ were compared on the basis of real experimental data for the Paramecium caudatum population. The calculation results were as follows:

Model	**error**
Classical model, $\alpha = 1$	51.37%
Fractional model, $\alpha = \frac{1}{2}$	43.63%
Fractional model, $\alpha = \frac{1}{3}$	38.08%

Therefore, for the real data, the fractional model of order $\alpha = \frac{1}{3}$ gave the smallest error. This shows that taking into account the memory effect in the population process by means of a fractional-order derivative increases the accuracy of the model.

At the same time, since the population in the real data stabilized around $58-60\, cells / mL$, it was determined that simple exponential and fractional exponential models are not fully sufficient. To obtain more accurate results in future studies, it is advisable to use a fractional-order logistic model.

As the order of the fractional derivative decreases, the memory effect in the model becomes stronger. That is, the current growth of the population depends not only on the current value but also more strongly on the states of previous periods. Therefore, for smaller values of the derivative order, population growth proceeds

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more slowly than in the classical exponential model. Biologically, this means that food shortage, environmental influence, disease, or external factors in previous periods affect subsequent growth for a longer time.

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